

Variograms and their properties

Spatial Statistics

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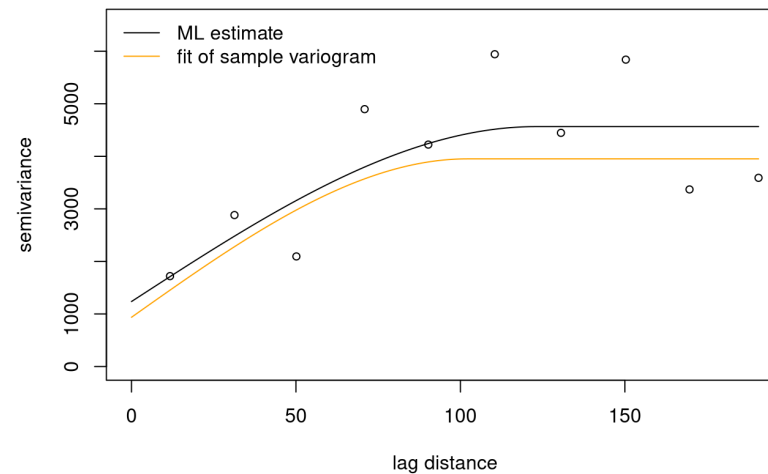
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1 Variogram function

i Variogram

A variogram is a mathematical function that measures how much data points differ from each other based on the distance and direction separating them.

- y-axis: semivariance (to measure the similarity of observations)
- x-axis: lag (relative distance between points, grouped by bins)
- Depending on the form of the variogram function the spatial prediction map will have different properties.
- Variogram function is a user choice.

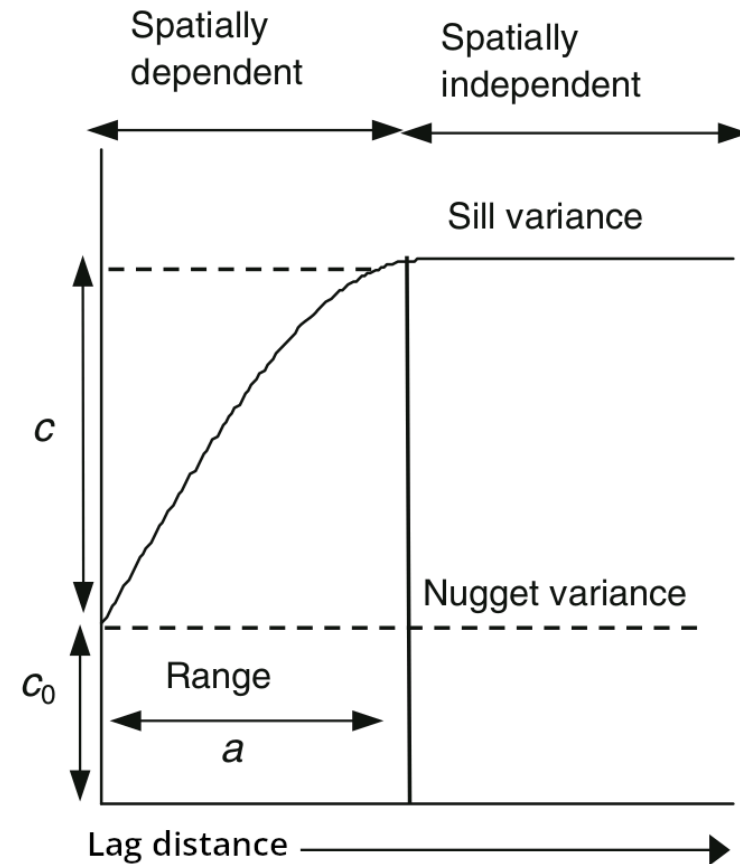


Variogram Wolfcamp data, from previous chapter 4

Variogram function structure

Principle features of the variogram function

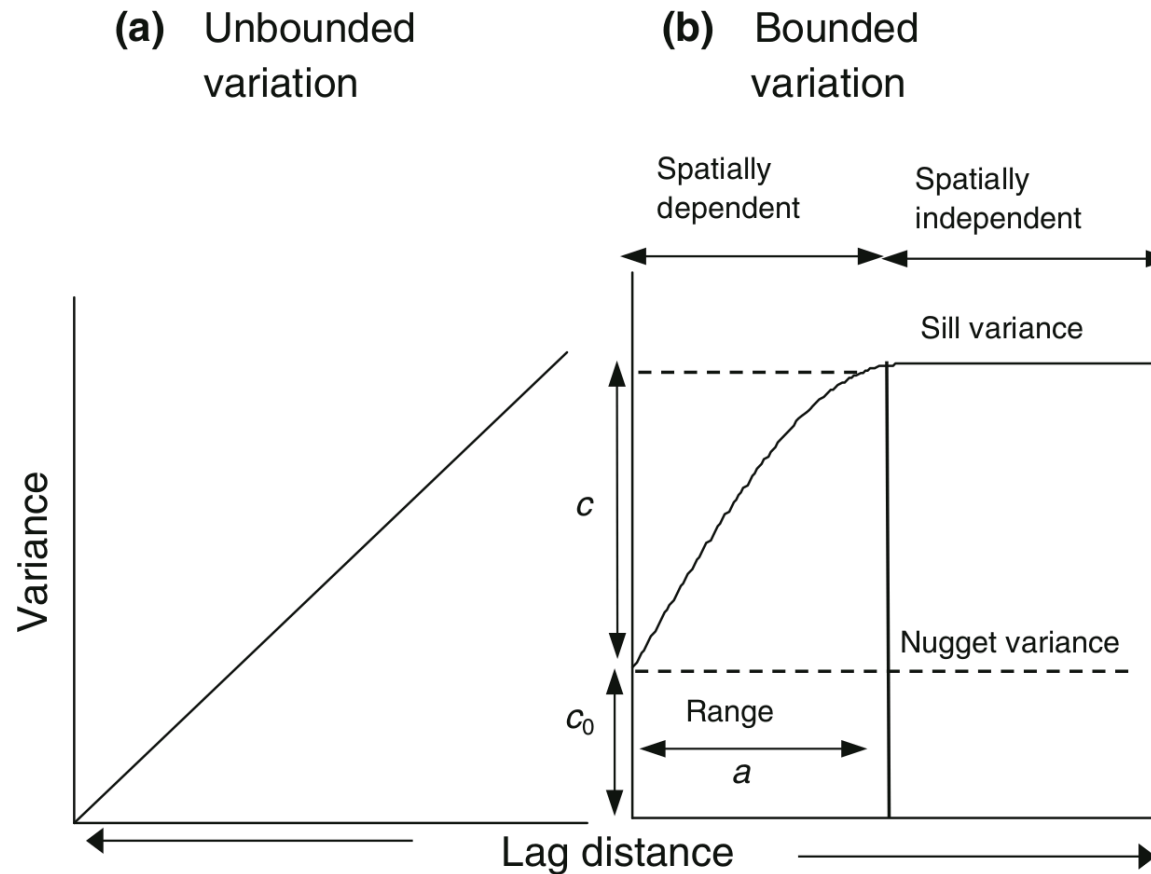
- Increase in variance with increasing lag
- Function must guarantee non-negative variances
- **Sill** variance $c_0 + c$, i.e. an upper bound
- **Range** of spatial correlation a , where auto-correlation becomes 0
- **Nugget** variance c_0 , i.e. a positive intercept or offset to account for smallest scale variability or measurement error
- with c often called **partial sill**, i.e. part of variance with spatial structure up to a



(Oliver and Webster, 2015, Fig. 3.10-b)

Side note: special cases

- *Anisotropy*, i.e. directional variation depending on angle
- *Unbounded* variogram (not second-order stationary)
- *Pure Nugget* variogram -> no spatial auto-correlation



(Oliver and Webster, 2015, Fig. 3.10)

Pure nugget variogram

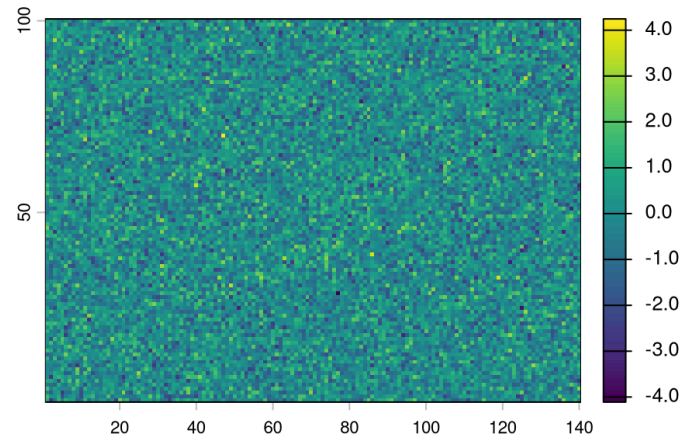
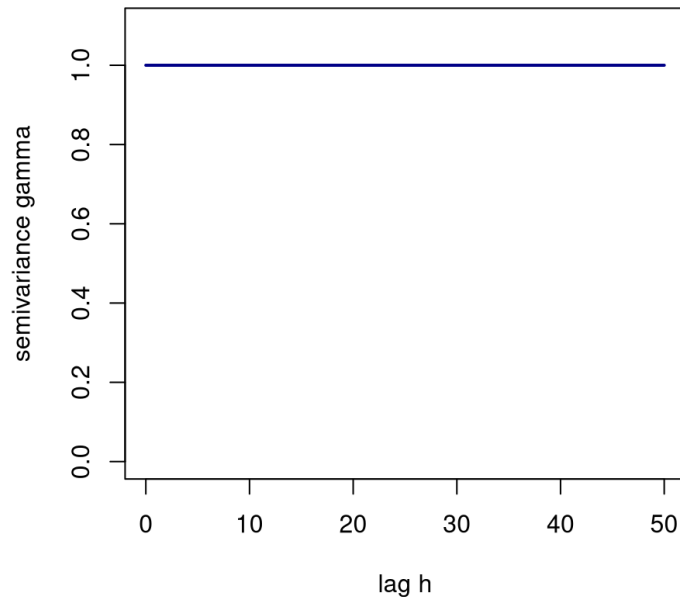
Absence of auto-correlation (nugget effect covariance)

$$V(h) = \begin{cases} 0 & \text{if } h = 0 \\ c_0 & \text{otherwise} \end{cases}$$

Mechanism: measurement error and/or small-scale spatial variation

Pure nugget variogram – realization

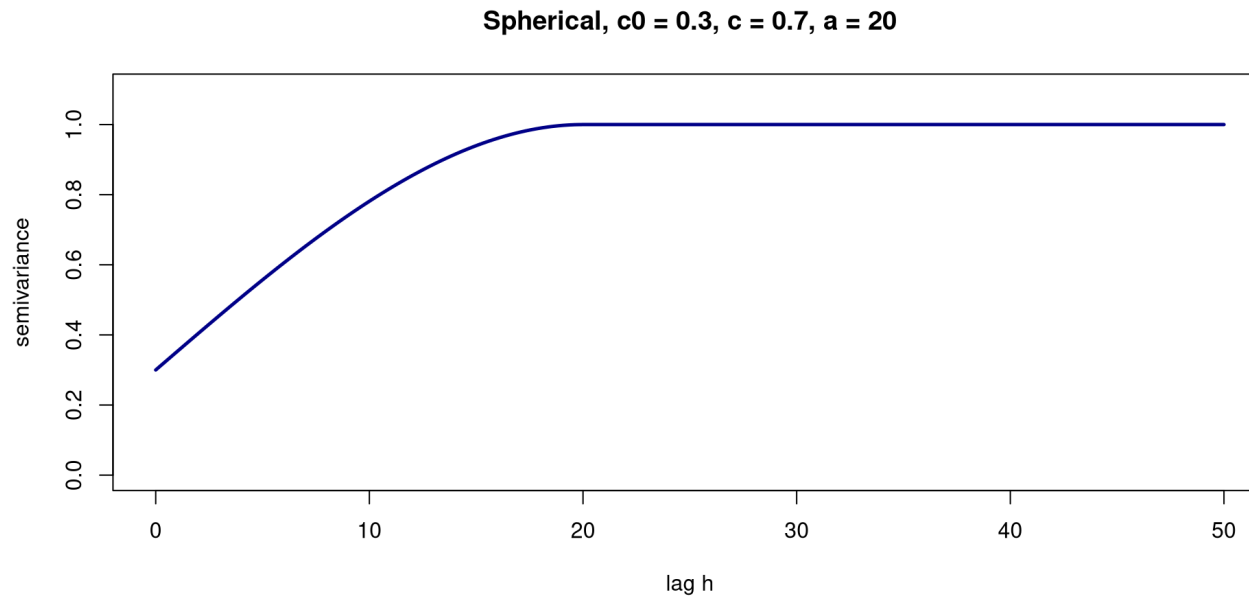
```
1 library(sp); library(terra); library(gstat)
2 # unconditional Gaussian simulation on a grid
3 xy <- expand.grid(1:140, 1:100)
4 names(xy) <- c("x","y")
5 gridded(xy) = ~x+y
6 v.m <- vgm(nugget=1, model = 'Sph', psill = 0, range = 0.0001)
7 plot(variogramLine(v.m, maxdist = 50), type = "l", ylim = c(0,1.1))
8 g.sim <- gstat(formula = z~1, dummy = TRUE, beta = 0, model = v.m, nmax = 100)
9 r.sim <- predict(g.sim, xy, nsim = 1)
10 plot(rast(r.sim))
```



Spherical variogram function

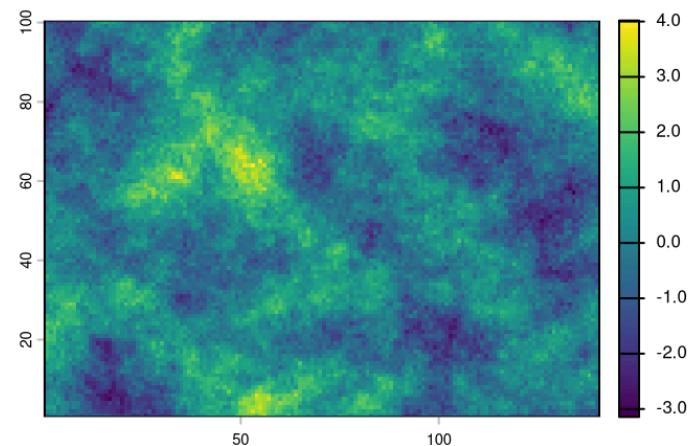
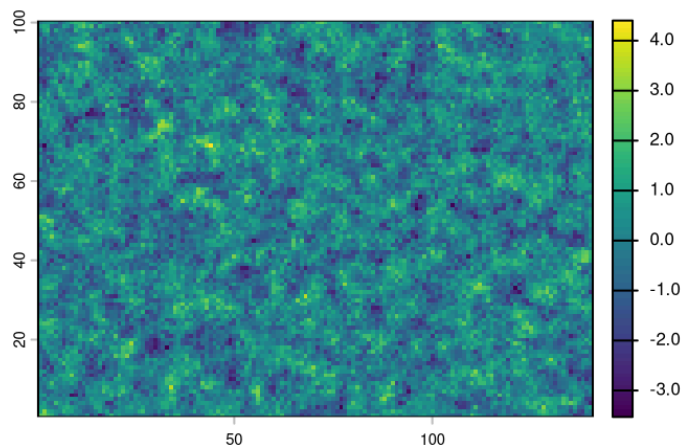
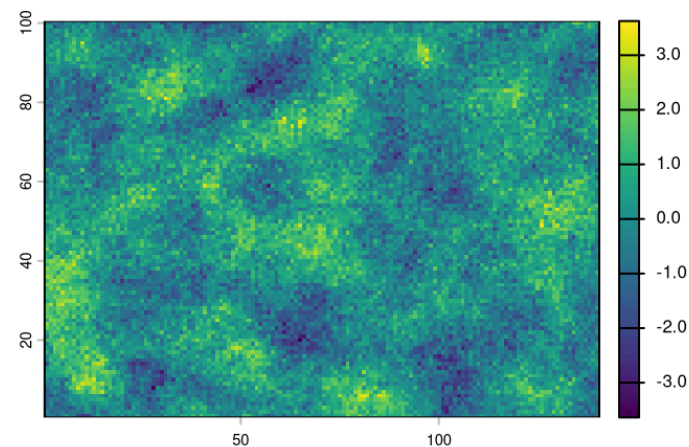
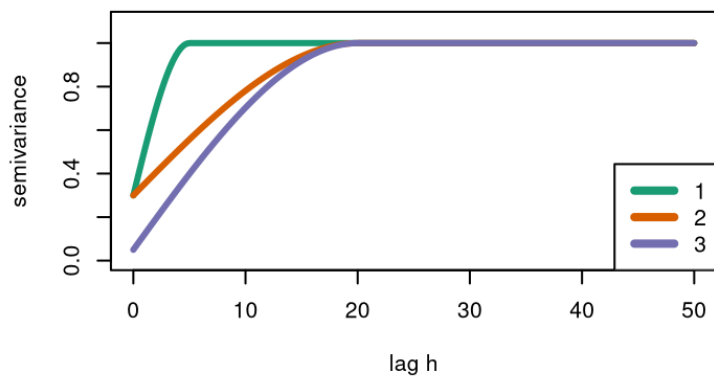
$$V(h) = \begin{cases} c_0 + c \left\{ \frac{3h}{2a} - \frac{1}{2} \left(\frac{h}{a} \right)^3 \right\} & \text{for } 0 < h \leq a \\ c_0 + c & \text{for } h > a \\ c_0 & \text{for } h = 0 \end{cases}$$

where c_0 is the nugget variance, c the variance of spatially correlated component and a is the range of spatial dependence.



Spherical variogram function – realizations

```
1 A <- vgm(nugget=0.3, model = 'Sph', psill = 0.7, range = 20)
2 B <- vgm(nugget=0.05, model = 'Sph', psill = 0.95, range = 20)
3 C <- vgm(nugget=0.3, model = 'Sph', psill = 0.7, range = 5)
```

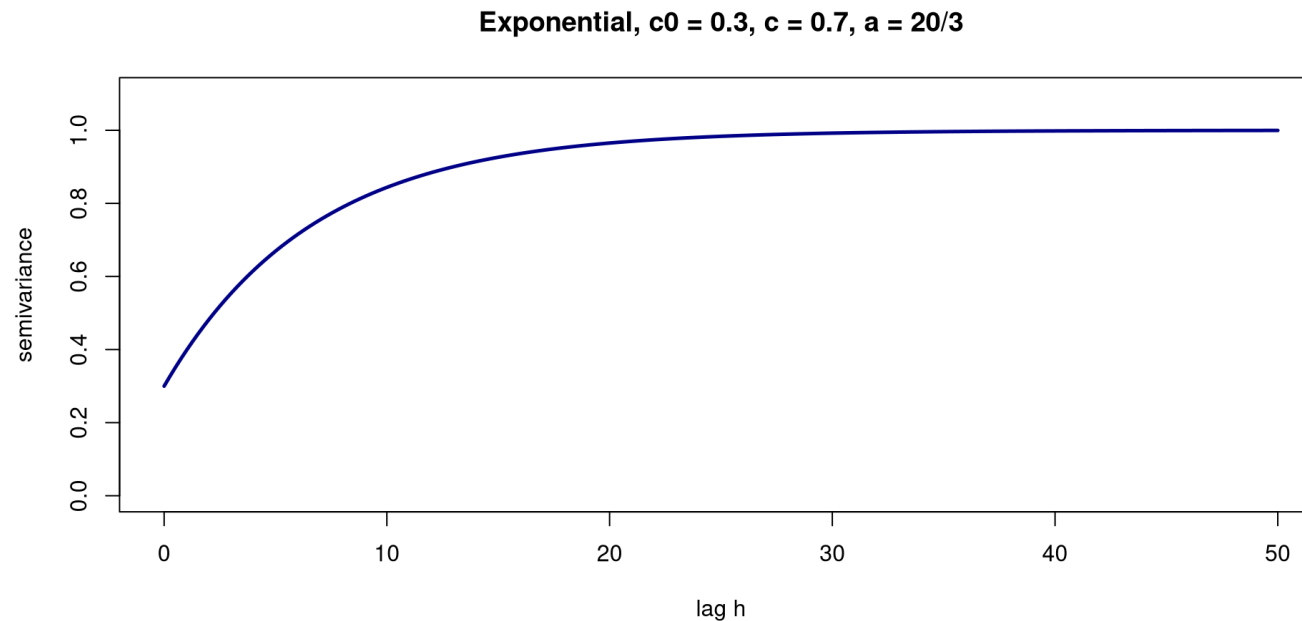


Exponential variogram function

$$V(h) = \begin{cases} c_0 + c \left\{ 1 - \exp\left(-\frac{h}{a}\right) \right\} & \text{for } 0 < h \\ c_0 & \text{for } h = 0 \end{cases}$$

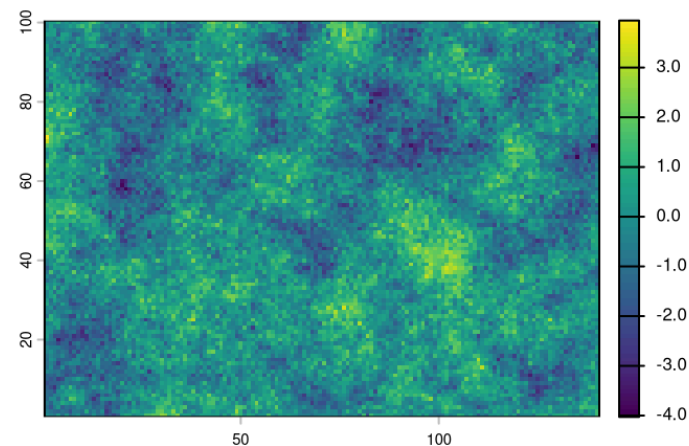
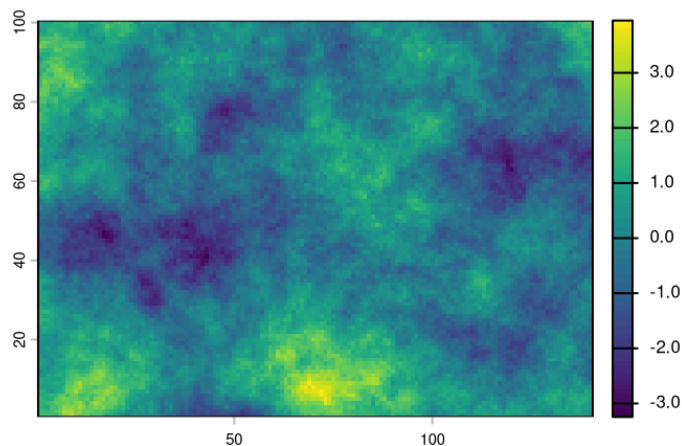
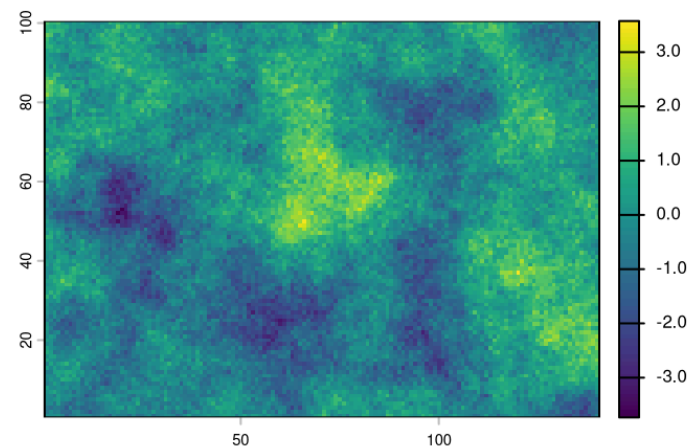
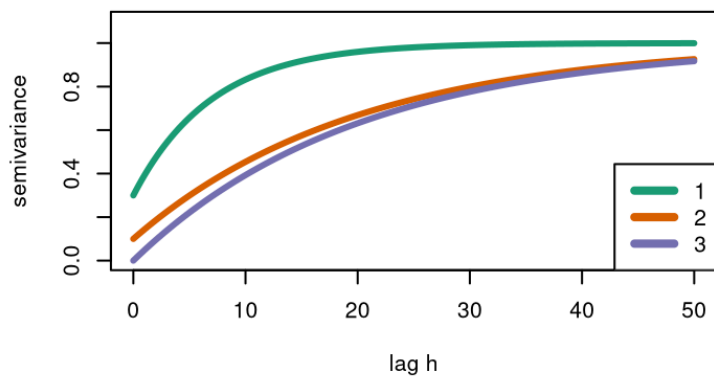
with a here being a distance parameter.

The function approaches the sill asymptotically and does not have a finite range. For practical purposes usually an effective range a' is used which is approximately $3a$.



Exponential variogram function – realizations

```
1 A <- vgm(nugget=0.3, model = 'Exp', psill = 0.7, range = 7)
2 B <- vgm(nugget=0.0001, model = 'Exp', psill = 0.999, range = 20)
3 C <- vgm(nugget=0.1, model = 'Exp', psill = 0.9, range = 20)
```



Matérn (Gaussian) variogram function

$$V(h) = c_0 + c \left\{ 1 - \frac{1}{2^{v-1}\Gamma(v)} \left(-\frac{h}{a} \right)^v K_v \left(\frac{h}{a} \right) \right\}$$

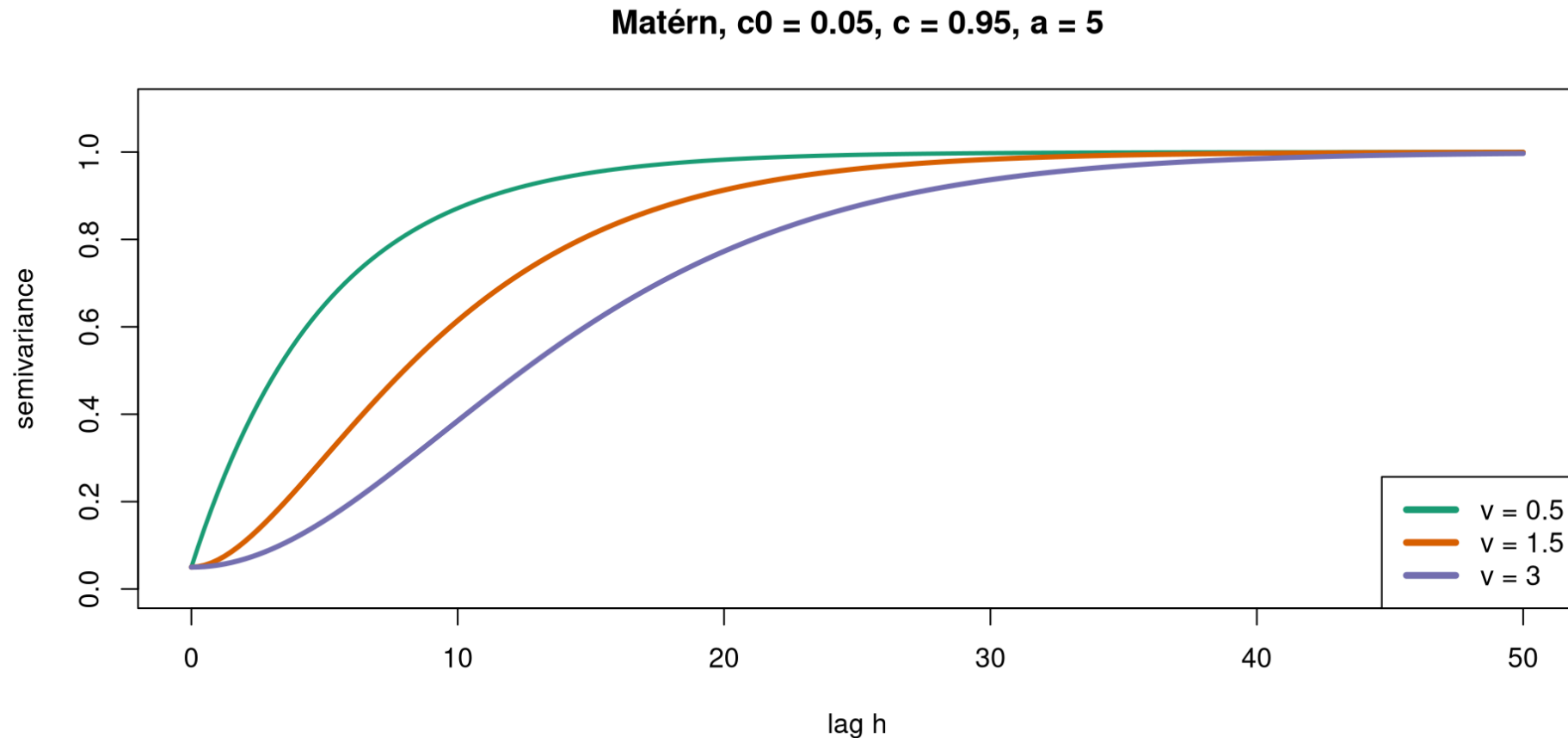
with c_0 , c and a being nugget, sill and range parameters as before.

The equation embodies the gamma function Γ with parameter v and the Bessel function of the second kind, K_v for parameter v .

Parameter v describes the smoothness of variation and can vary from 0 (very rough) to infinity (very smooth).

With $v = 0.5$ it becomes the exponential variogram.

Matérn function – Variogram smoothness

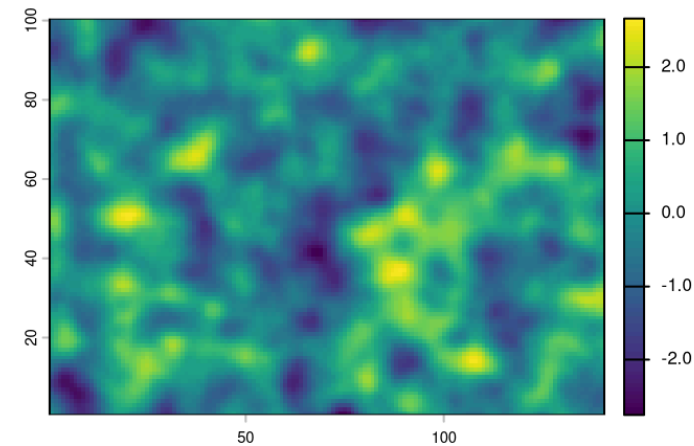
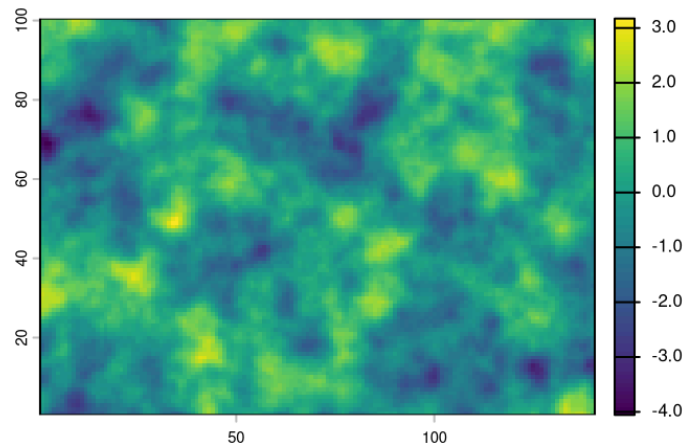
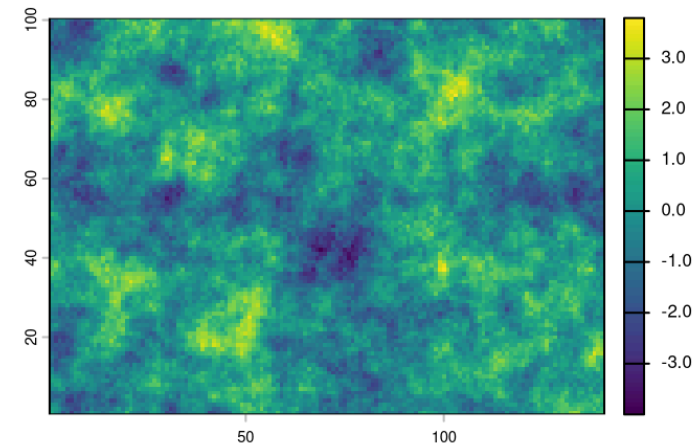
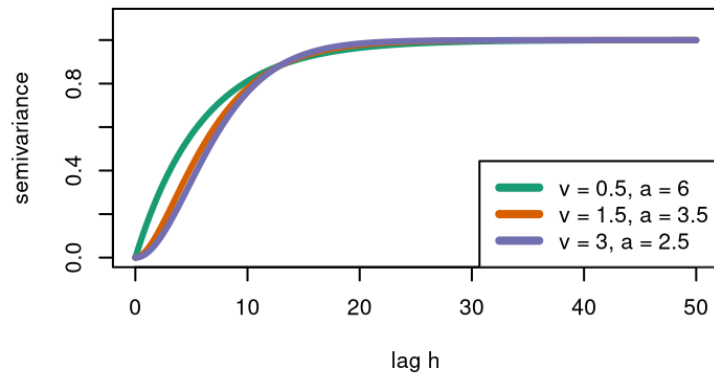


Shape of variogram close to origin controls smoothness of realizations of stochastic processes:

1. Variogram with nugget: realizations non-continuous
2. Variogram grows linearly at origin: realizations continuous, but not everywhere differentiable
3. Variogram grows at at least quadratically at origin: realizations everywhere at least once differentiable

Matérn variogram function – realizations

```
1 A <- vgm(nugget=0, model = 'Mat', psill = 1, range = 6, kappa = 0.5)
2 B <- vgm(nugget=0, model = 'Mat', psill = 1, range = 3.5, kappa = 1.5)
3 C <- vgm(nugget=0, model = 'Mat', psill = 1, range = 2.5, kappa = 3)
```





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