

Theory: Stochastic process

Spatial Statistics

Madlene Nussbaum

m.nussbaum@uu.nl

Utrecht University, Department of Physical Geography

Overview

Chapters 6-12

Geostatistics: theory, variogram properties, maximum likelihood, prediction

- Stochastic process
- Realization of stochastic process
- Variogram functions
- Sample variogram and fitting of variogram function
- Maximum likelihood estimation of model parameters
- Model selection/inference
- Predictions for ordinary and universal/external drift kriging

1 Stochastic process

Terminology and model notation (chapter 4)

Model for data: $Y_i = S(\mathbf{x}_i) + Z_i$

where

Y_i : i^{th} datum

$S(\mathbf{x}_i)$: "signal" (= true quantity) at location $\mathbf{x}_i$

Z_i : iid. random measurement error

Decomposition of signal into trend $\mu(\mathbf{x}_i)$ and stochastic fluctuation:

$$S(\mathbf{x}_i) = \mu(\mathbf{x}_i) + E(\mathbf{x}_i)$$

where commonly a linear model is used for $\mu(\mathbf{x}_i)$

$$\mu(\mathbf{x}_i) = \sum_k d_k(\mathbf{x}_i) \beta_k = \mathbf{d}(\mathbf{x}_i)^T \boldsymbol{\beta}$$

with $d_k(\mathbf{x}_i)$ denoting (spatial) covariates and $\{E(\mathbf{x}_i)\}$ a zero mean stochastic process (random field).

Realization of stochastic process

- Spatial phenomena obey laws of physics $\Rightarrow$ are deterministic, have physical causes
- Numerous processes and interactions thereof produce current complex outcome
- Variation appears random $\Rightarrow$ in geostatistical terms viewed as a random process
- e.g. rainfall pattern, soil properties, air pollution as a realization of a random process
- Each location x is associated with a suite of values with a known distribution
- Actual value observed at x is viewed as one value of this distribution, allocated at random
- Random function has no mathematical description, but “structure” in the sense of correlation in space (or time) and describes relation of random values at different locations to each other

i Spatial stochastic process (random process)

$\{S(\mathbf{x})\}$: Collection (= set) of random variables $S(\mathbf{x}) : \mathbf{x} \in D \subset \mathbb{R}^d$, at location x in area D , with a well defined joint distribution

Stationary and isotropic stochastic processes

Stationarity: Assumption that allows to treat data with same degree of variation over region of interest.

Strictly stationary process: Joint distributions of arbitrary collections of random variables $\{S(\mathbf{x}_1), \dots, S(\mathbf{x}_n)\}$ are invariant to translations by vector $\mathbf{h} \in \mathbb{R}^d$

$\{S(\mathbf{x}_1), \dots, S(\mathbf{x}_n)\}$ and $\{S(\mathbf{x}_1 + \mathbf{h}), \dots, S(\mathbf{x}_n + \mathbf{h})\}$ have the same joint distribution:

$$F(s_1, \dots, s_n; \mathbf{x}_1, \dots, \mathbf{x}_n) = F(s_1, \dots, s_n; \mathbf{x}_1 + \mathbf{h}, \dots, \mathbf{x}_n + \mathbf{h})$$

Isotropic: Weakly stationary process that is invariant to rotations (opposite: anisotropic).

Gaussian stochastic process: All joint and conditional distributions are normal.

Second-order stationary stochastic processes

Weakly or second-order stationary process:

Distributions of arbitrary pairs of random variables $(S(\mathbf{x}), S(\mathbf{x} + \mathbf{h}))$ satisfy:

1. $\mathbb{E}[S(\mathbf{x})] = \text{constant}$ (independent of $\mathbf{x}$)
2. $\text{Cov}(S(\mathbf{x} + \mathbf{h}), S(\mathbf{x})) = \gamma(\mathbf{h})$ (independent of $\mathbf{x}$)
3. $\text{Var}(S(\mathbf{x})) = \text{constant}$ (independent of $\mathbf{x}$)

⇒ Covariance depends on h and only on h , the separation between samples in both distance and direction

⇒ Strict stationarity implies weak stationarity.

⇒ Stationarity is required for estimation/prediction with a single realization of the stochastic process.

2 Covariance function and variogram

Definition of variogram $V(\mathbf{h})$ and covariance function $\gamma(\mathbf{h})$:

$$V(\mathbf{h}) = \frac{1}{2} \text{Var} (S(\mathbf{x} + \mathbf{h}) - S(\mathbf{x}))$$

$$\gamma(\mathbf{h}) = \text{Cov} (S(\mathbf{x} + \mathbf{h}), S(\mathbf{x}))$$

Relation between variogram and covariance function:

$$V(\mathbf{h}) = \gamma(0) - \gamma(\mathbf{h}), \quad \text{with} \quad \gamma(0) = \text{Var}(S(\mathbf{x}))$$

⇒ Variogram is preferred.

Variogram is based on differences only (no distribution mean as in covariance). Allows to relax stationarity assumptions even further, instead of $\mathbb{E}[S(\mathbf{x})] = \text{constant}$, we can assume (intrinsic stationarity):

$$\mathbb{E}[S(\mathbf{x} + \mathbf{h}) - S(\mathbf{x})] = 0$$

Variogram and correlogram

Relation between correlogram and covariance function for weakly stationary process:

$$\rho(\mathbf{h}) = \frac{\gamma(\mathbf{h})}{\gamma(0)}$$

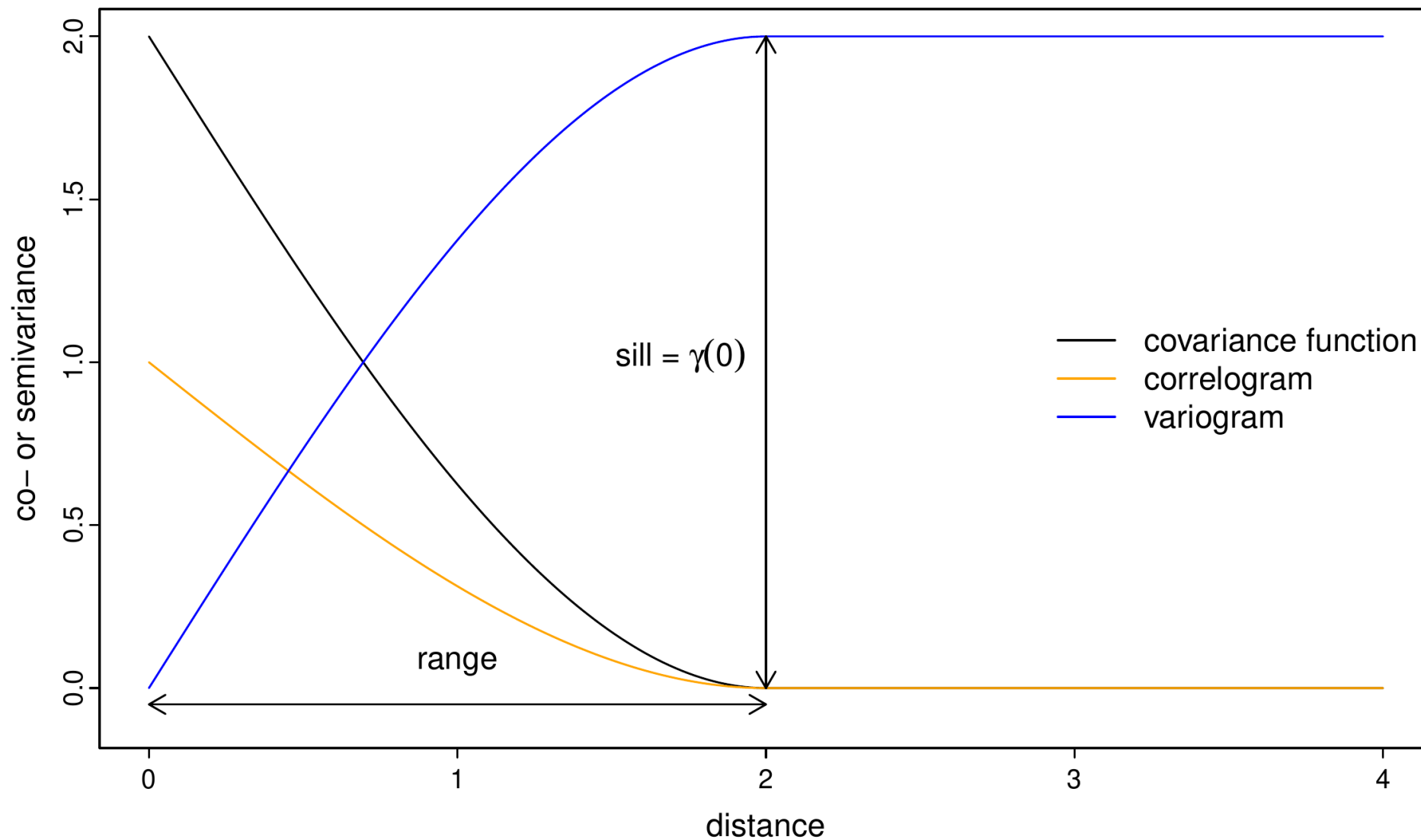
Relation between variogram and correlogram:

$$V(\mathbf{h}) = \gamma(0)(1 - \rho(\mathbf{h}))$$

Symmetry:

$$V(\mathbf{h}) = V(-\mathbf{h}), \quad \gamma(\mathbf{h}) = \gamma(-\mathbf{h}), \quad \rho(\mathbf{h}) = \rho(-\mathbf{h})$$

Covariance and Variogram Plot



Summary stochastic process

- Stochastic process: generalisation of multidimensional random variable
- Stationarity assumption required for estimation from single realisation of stochastic process
- In practice assumption of weak stationarity:
 1. constant mean
 2. constant variance
 3. covariance and semivariance depends only on lag distance but not on location
- Often additional assumption of isotropic auto-correlation



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